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Review of the Ph.D. Thesis
„Topology of configuration spaces for particles on graphs",
submitted by Tomasz Maciazek

Since the work of Souriau and Leinaas & Myrheim, it has been understood that quantum mechanics on multiply connected spaces admits different types of particle statistics, which are classified by the conjugacy classes of unitary representations of the fundamental group. Motivated by this fact, the present thesis undertakes a study of configuration space topology for particles in the simplified setting of graphs.

The first part of the thesis (Chapters 1-3) is a comprehensive review of textbook material from the mathematical area of topology. After an exposition of the main ideas underlying quantum kinematics (Chapter 1), the thesis begins (in Chapter 2) by introducing the basic notions of *CW*-complex, fundamental group of a topological space, homology and cohomology of chain complexes, exact sequences, and Künneth formula. The thesis continues (in Chapter 3) with foundational material on the topology of vector bundles, including the notion of universal bundle, Chern characteristic classes, and the Chern character map from topological *K*-theory to rational cohomology. For later use in the thesis, particular attention is paid to flat bundles and their moduli spaces.

After introducing some elements of graph theory, Chapter 4 formulates two discrete models (actually, deformation retracts) of the configuration spaces of n identical particles on a graph, namely that of Abrams and that of Swiatkowski. Some definitions and results for the first homology group of such spaces are also reviewed.

The main part of the thesis is Chapter 5. There, homology groups are computed for a selection of graph configuration spaces, using the discrete models of Adams and Swiatkowski. The main tool is Forman's version of Morse theory for cell complexes. By iterating *ad infinitum* the flow of a so-called discrete vector field, Forman obtains a Morse chain complex as the limit space spanned by flow-invariant chains. The homology groups are then inferred from the boundary map on the Morse complex, expressed as a matrix in the basis of critical cells. The kernel and elementary divisor of that matrix are computed numerically using standard library routines. An alternative tool is to proceed analytically, by exploiting Mayer-Vietoris long exact sequences (for the *CW*-complex divided into sub-complexes) in homology.

The numerical and analytical approach of Chapter 5 is applied to graphs of connectivity one in Ch. 5.3, to wheel graphs in Ch. 5.4, and to a few complete bipartite graphs in Ch. 5.5 and 5.6. The last section 5.7 proves a theorem that states the conditions under which the second homology group is generated by product cycles of the graph configuration space.

In total, this is a substantive thesis using state-of-the-art techniques to obtain some original results. The results presented in Ch. 5.3 are mostly from a joint publication (with A. Sawicki) in the Journal of Mathematical Physics. The results of the thesis support a recent conjecture by E. Ramos, stating that the Betti numbers of graph configuration spaces exhibit the stability phenomenon of becoming regular above a certain value of the particle number.

Let me finish with a few critical remarks. First of all, the Chapters 1, 2, and 3 on mathematical background are quite expansive, and they introduce a lot of material that is never used in the remainder of the thesis. In contrast, Chapter 4 (containing less elementary material) is very condensed and difficult to follow (for me as a non-expert anyway). In particular, Lemma 4.2, Lemma 4.3, and Theorem 4.4 are stated without proof, and no source is given.

Second, I also do not appreciate the opening remark (in Chapter 1: Introduction) that "The first part of this thesis … … is an attempt to collect and organize some of the results that are partially a folklore knowledge in mathematical physics". Is the author insinuating that mathematical physicists know the textbook results quoted in Chapters 1-3 only as folklore?

(Or the textbook results are just folklore and in need of rigorous justification by the thesis author?)

In some places, the thesis is written in a hasty or sloppy manner. For example, the first Betti number below Definition 4.3 is quoted as $E(\Gamma) - V(\Gamma) + N$ (the author does not bother to distinguish between a set and the cardinality of the set).

In 2.2.1 (“Homology of chain complexes”) the author writes “The first step in defining a chain complex … is choosing an orientation for each cell”. I disagree. To define a vector space, one does not need to choose a basis for it. Similarly, to define a chain complex and its invariantly defined boundary operator, no choice of basis by oriented cells is needed.

In the opposite vein, the differential operator Δ introduced in Eq. (2.8) is not invariantly defined, as the “standard” pairing used there is not standard but depends on a fixed choice of generators.

My final complaint is that the author does not understand how to use punctuation by commas in the English language. This made my reading somewhat tiring at times.

In the balance, my opinion is that the submitted thesis does have the scientific substance required of a doctoral thesis. I therefore recommend that the candidate be admitted to the next steps of the doctoral procedure.

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