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Report on the thesis

UNIVERSALITY IN QUANTUM COMPUTATION

by Katarzyna Karnas

The title of the thesis refer to an issue that arises when one (imagines that one) builds a quantum computer operating with a discrete and finite set of unitary transformations, called 'gates' in this context, and asks whether using this set one can approximate arbitrary unitary transformations to some given precision ϵ, and if so how the number of discrete gates that one needs grows when ϵ shrinks. The first question is that of universality, the second that of efficiency. Since the sets of gates one is likely to choose in quantum computation are likely to be rather special sets (because of the demands of error correcting codes), it becomes important to be able to check if a given set fails to be universal. These questions clearly arise naturally also in mathematics. Applications to control theory and to other areas of physics can easily be imagined. In other words, the topic is a good one.

After a brief introductory chapter, and a chapter covering the mathematical background, the thesis contains two chapters containing original results. I think it is fair to say that the approach taken in chapter three, based on methods from algebraic number theory, is shown to run into excessively complex calculations. The group theoretical approach taken in chapter four on the other hand is quite successful. I will comment on these chapters in turn.

Chapter 1: The introduction comes quickly to the point, and gives the formal version of the first sentence in this report, ably and well. The thesis considers a quantum computer operating on qudits (as opposed to qubits only), so the question is whether every unitary $g \in SU(d)$ can be approximated by a 'word' formed from a finite set of group elements. An appropriate set of references is given. I would perhaps have liked to see another half a page explaining why this is all one needs to do, in order to approximate arbitrary unitaries in $SU(\mathcal{H}_1 \otimes \ldots \mathcal{H}_k)$.

Chapter 2: This chapter covers a huge amount of ground, including the necessary number theory, algebraic number theory, and Lie groups. It ends with an account of the exciting topic of spectral gaps of averaging operators, which looks likely to be the key to the question about efficiency. Proofs of mathematical theorems are given when the reader actually needs to know the proofs, and not only the statements, in order to follow the arguments in the rest of the thesis (as is the case for instance for the distance inequalities obeyed by group commutators). Also, the length of the explanations grow when the reader is likely to need them (as in the discussion of spectral gaps).

Perhaps this is also the place to point out that the thesis contains quite a few misprints ('diffential manifolds') and even small slips in some formulations (eg. uniqueness should not be part of the definition of a minimal polynomial, and the sentence immediately following Definition 2.15 makes clear that the author knows this). However, as Newton said, "such errors as do not depend on wrong reasoning can be of no great consequence and can be corrected by the reader". The main comment I have on this chapter is that is is organized in a quite intelligent way.

Chapter 3: This chapter develops a previously proposed method to decide whether a given set of gates is universal. It relies on a theorem by Schur (Theorem 2.5 in the thesis), which says that a finitely generated matrix group has an infinite number of elements if and only if it contains infinite order elements, and then uses elements of algebraic number theory to recognize any infinite order element that may occur. The reason that number theory enters is that it becomes necessary to ask whether rotations through irrational angles will occur for some group elements. The method is applied to two interesting examples of sets of qubit gates.

In my first paragraph I hinted that this chapter seems to me to be in a sense less fruitful than the next. The point is that the necessary calculations turn out to be very complex. The reader is left with the feeling that the number of examples that can actually be treated with this method is going to be very limited. But this also is useful information, so that the chapter constitutes a real contribution to the subject.

Chapter 4: If we are given a finite set of gates and form 'words' of arbitrary lengths from them, the question of universality is the question whether the closure of the set of words gives us all of $SU(d)$, or whether one ends up with a finite or infinite subgroup of $SU(d)$. What we need is an algorithm that can be applied to any finite set, and give us the answer to this question. We also want a guarantee that the algorithm returns the answer in a finite number of steps. In this chapter the author proposes such an algorithm, and shows that it has the desired property. The calculations needed involve matrix multiplication and the solution of algebraic equations, and it has been implemented in the form of a computer program. This is clearly a real contribution to the subject.

One highlight of the chapter is the introduction of the maximal exponent N_G. The algorithm relies on the criterion that the set of gates will be universal if it can be used to construct unitaries that are close the centre of $SU(d)$, without actually belonging to the centre. Mathematicians know that there always exists an integer n such that, given a group element g, g^n lies in a suitable ball around a central element. The maximal exponent N_G is the largest such exponent that can occur for some group element. In the thesis an upper bound for N_G is calculated. Then numerical evidence is reported suggesting that the bound is not sharp, except for $SU(2)$. I bring this up as an example of an interesting open question raised in the thesis—good work always leads to open questions.

The thesis is based on three papers published together with the supervisor, and as usual the question about the author's share of the work comes up. It is a question that is hard to answer from a distance. However, based on the mature and personal way that the thesis is written in, and also on an afternoon's discussion with the author (in which I asked her to explain various difficult points), I feel sure that she is completely on top of all the material. *I therefore have the pleasure to recommend, with no hesitation whatsoever, that the thesis should be publicly defended.*

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